2.2 Quadratic Functions

Quadratic functions have two forms:

1. $f(x) = ax^2 + bx + c$ The x– coordinate of the vertex is: $x = \frac{-b}{2a}$

2. $f(x) = a(x - h)^2 + k$ The vertex is: V(h, k).

Characteristics of the Graph of the Quadratic Function:

1. The graph of the quadratic function is called a parabola.
2. The vertex could be the highest point of the graph, or the lowest point of the graph.
3. The parabola opens up if a is positive, and it opens down if a is negative.
4. The vertical line that passes through the vertex is called the axis of symmetry.

Identify the vertex, the equation of the axis of symmetry, the x and y– intercepts of the parabola. Find the domain and range of the function.

1) $f(x) = x^2 - 4x - 5$ 1) ____________

Solution: $a = 1$, $b = -4$, $c = -5$

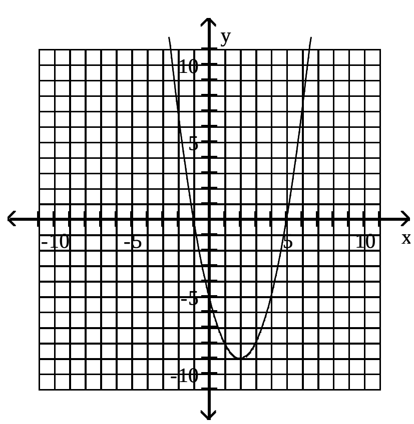
Vertex: $x = \frac{-b}{2a} = \frac{4}{2} = 2.$ $y = 4 - 8 - 5 = -9.$ V(2, –9)

Axis of Symmetry: $x = 2$

y–intercept: set $x = 0$, then $y = -5$

x–intercept: set $y = 0$, then $x^2 - 4x - 5 = 0$

$\rightarrow$ $(x - 5)(x + 1) = 0$ $\rightarrow$ $x = 5$ and $x = -1$



Domain: $(-\infty , \infty)$ Range: $[-9 , \infty)$

Identify the vertex, the equation of the axis of symmetry, the x and y– intercepts of the parabola. Find the domain and range of the function.

2) $y = -(x - 1)^2 + 4$ 2) ___________

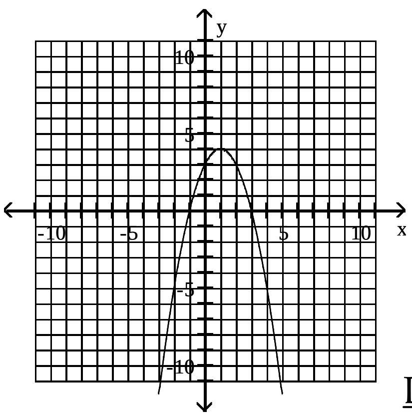
Solution: $a = -1$, $h = 1$, $k = 4$

Vertex : V(1, 4) Axis of Symmetry: $x = 1$

y–intercept: set $x = 0$, then $y = -1 + 4 = 3$

x–intercept: set $y = 0$, then $-(x - 1)^2 + 4 = 0 \rightarrow (x - 1)^2 = 4$

$\rightarrow x - 1 = 2$ or $x - 1 = -2 \rightarrow x = 3$ or $x = -1$



Domain: $(-\infty , \infty)$ Range: $(-\infty , 4]$

Determine the x and y– intercepts of the parabola.

3) $f(x) = -4x^2 - 5x + 6$ 3) ___________

Solution:

y–intercept: set $x = 0$, then $y = 6$

x–intercept(s): set $y = 0$, then $-4x^2 - 5x + 6 = 0$

$\rightarrow -1(-4x^2 - 5x + 6 = 0) \rightarrow 4x^2 + 5x - 6 = 0$

$\rightarrow (4x - 3)(x + 2) = 0$

$\rightarrow 4x - 3 = 0$ or $x + 2 = 0 \rightarrow x = \frac{3}{4}$ or $x = -2$

Maximum and Minimum of Quadratic Functions

1. If $a < 0$, then f has a maximum that occurs at the x-value of the vertex.
2. If $a > 0$, then f has a minimum that occurs at the x-value of the vertex.

Solve the problem.

4) The total revenue from the sale of x televisions is given by $R(x) = 60x - 3x^2$, how many televisions must be sold that will yield a maximum revenue? 4) ____________

Solution: $R(x) = -3x^2 + 60x$ $\quad a = -3, b = 60$

Revenue is a maximum at $x = \frac{-b}{2a} = \frac{-60}{-6} = 10.$

So, 10 televisions must be sold to yield a maximum revenue.

2.2 Exercises pg 313 (17, 31, 65, 100) (26, 27, 66, 101)

2.4 Dividing Polynomials; Remainder and Factor Theorems

Division Algorithm:

Dividend = Divisor • Quotient + Remainder is the same as $f(x) = (x - k)Q(x) + R$.

Use synthetic division to perform the division.
Express the answer in the form $f(x) = (x - k)Q(x) + R$.

5) $\frac{5x^3 - 13x^2 - x + 10}{x + 2}$ 5) ____________

Solution:

−2)	5	−13	−1	10
		−10	46	−90
	5	−23	45	−80

$f(x) = (x + 2)(5x^2 - 23x + 45) - 80$

Remainder Theorem: If f(x) is divided by x – k, then the remainder is equal to f(k).

Use the remainder theorem and synthetic division to find f(k).

6) $k = -3$; $f(x) = 3x^3 - 8x^2 - 3x + 24$ 6) ____________

Solution:

−3)	3	−8	−3	24
		−9	51	−144
	3	−17	48	−120

$f(-3) = R = -120$

Zero Theorem: x = k is a zero of f(x) if the remainder is zero.

Use synthetic division to decide whether the given number is a zero of the given polynomial.

7) 2 ; $f(x) = x^4 + 5x^2 - 32$ 7) ____________

Solution:

2)	1	0	5	0	−32
		2	4	18	36
	1	2	9	18	4

Since $R \neq 0$, then x = 2 is not a zero of f(x).

Factor Theorem: (x – k) is a factor of f(x) if x = k is a zero of f(x).

Use the factor theorem to decide whether or not the second expression is a factor of the first.

8) $f(x) = x^3 - 4x^2 + 9x - 10$; $x - 2$ 8) ____________

Solution:

2)	1	−4	9	−10
		2	−4	10
	1	−2	5	0

→ x = 2 is a zero of f(x) → x – 2 is a factor of f(x)

Factor f(x) into linear factors given that k is a zero of f(x). Find all other zeros.

9) $f(x) = 2x^3 - 3x^2 - 5x + 6$; $k = 1$ 9) ____________

Solution:

$$\begin{array}{r|rrrr} 1 & 2 & -3 & -5 & 6 \\ & & 2 & -1 & -6 \\ \hline & 2 & -1 & -6 & 0 \end{array}$$

$f(x) = (x - k)Q(x) + R$

$= (x - 1)(2x^2 - x - 6) = (x - 1)(2x + 3)(x - 2)$

$2x + 3 = 0 \rightarrow x = -\dfrac{3}{2}$ $x - 2 = 0 \rightarrow x = 2$

So, the remaining zeros are: $x = -\dfrac{3}{2}$ and $x = 2$

2.4 Exercises pg 343 (21, 35, 43) (22, 37, 46)

2.5 Zeros of Polynomial Functions

Rational Zeros Theorem: If $f(x) = ax^n + bx^{n-1} + c\,x^{n-2} + \ldots + dx + e$, then the possible rational zeros are given by : $\dfrac{\text{the factors of } e}{\text{the factors of } a}$.

Find the zeros of the polynomial function.

10) $f(x) = 2x^3 + 3x^2 - 8x + 3$ 10) ____________

Solution: The possible rational zeros are given by:

$\dfrac{\text{the factors of } 3}{\text{the factors of } 2} = \dfrac{\pm 1 \pm 3}{\pm 1 \pm 2}$

So, the possible rational zeros are: $\pm 1 \ \pm \dfrac{1}{2} \ \pm 3 \ \pm \dfrac{3}{2}$.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$ $\rightarrow x = 1$ is a zero of f(x).

$f(x) = (x - k)Q(x) + R$

$= (x - 1)(2x^2 + 5x - 3) = (x - 1)(2x - 1)(x + 3)$

$2x - 1 = 0 \rightarrow x = \dfrac{1}{2}$ $x + 3 = 0 \rightarrow x = -3$

So, the zeros are: $x = 1$, $x = \dfrac{1}{2}$, and $x = -3$

11) $f(x) = x^3 - 2x^2 + 3x - 6$ 11) ___________

Solution:

The possible rational zeros are given by:

$$\frac{\text{the factors of } -6}{\text{the factors of } 1} = \frac{\pm 1 \;\; \pm 2 \;\; \pm 3 \;\; \pm 6}{\pm 1}$$

So, the possible rational zeros are: $\pm 1 \;\; \pm 2 \;\; \pm 3 \;\; \pm 6$.

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 0 \end{array} \rightarrow x = 2 \text{ is a zero of } f(x).$$

$f(x) = (x - k)Q(x) + R = (x - 2)(x^2 + 3)$

$x^2 + 3 = 0 \quad \rightarrow \quad x^2 = -3 \quad \rightarrow \quad x = \pm\sqrt{-3} = \pm i\sqrt{3}$

So, the zeros are: $x = 2$, $x = \pm i\sqrt{3}$

Conjugates Zeros Theorem:

If $a + bi$ is a zero of the polynomial function $f(x)$, then $a - bi$ is also a zero of $f(x)$.

Find a polynomial of lowest degree with only real coefficients and having the given zeros.

12) $n = 3$; 4 and $-6i$ are zeros ; $f(2) = 240$ 12) ___________

Solution:

$x = 4$ is a zero of $f(x)$ implies $(x - 4)$ is a factor of $f(x)$.

$x = -6i$ is a zero of $f(x)$, then $x = 6i$ is also a zero of $f(x)$.

implies $(x^2 + 36)$ is a factor of $f(x)$.

So, $f(x) = a(x - 4)(x^2 + 36)$

Since $f(2) = 240$, then $a(2 - 4)(4 + 36) = 240$

implies $-80a = 240 \rightarrow a = -3$

So, $f(x) = -3(x - 4)(x^2 + 36) = -3(x^3 - 4x^2 + 36x - 144)$

$= -3x^3 + 12x^2 - 108x + 432$

13) $n = 3$; -2 and $4 - i$ are zeros ; $f(1) = 60$ 13) ___________

Solution:

$x = -2$ is a zero of $f(x) \rightarrow (x + 2)$ is a factor of $f(x)$.

$x = 4 - i$ is a zero of $f(x) \rightarrow (x - 4 + i)$ is a factor of $f(x)$.

Using the Conjugate Zeros Theorem ;

$x = 4 + i$ is also a zero of $f(x) \rightarrow (x - 4 - i)$ is a factor of $f(x)$.

So, $f(x) = a(x + 2)(x - 4 + i)(x - 4 - i)$

$= a(x + 2)(x^2 - 4x - xi - 4x + 16 + 4i + ix - 4i - i^2)$

$= a(x + 2)(x^2 - 4x - 4x + 16 + 1)$

$= a(x + 2)(x^2 - 8x + 17)$

Since $f(1) = 60$, then $a(1 + 2)(1 - 8 + 17) = 60$

implies $30a = 60 \rightarrow a = 2$

So, $f(x) = 2(x + 2)(x^2 - 8x + 17) = (2x + 4)(x^2 - 8x + 17)$

$= 2x^3 - 16x^2 + 34x + 4x^2 - 32x + 68$

$= 2x^3 - 12x^2 + 2x + 136$

2.5 Exercises pg 356 (17, 20, 25) (19, 21, 27)

2.6 Rational Functions and Their Graphs

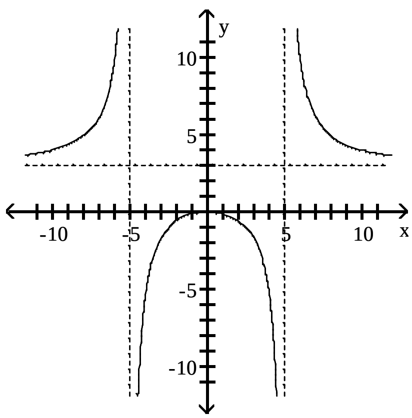
A rational function is of the form : $f(x) = \dfrac{P(x)}{Q(x)}$.

Vertical Asymptotes: The line x = a is a vertical asymptote of the graph of f(x) if $f(x) \to \infty$ or $-\infty$, as $x \to a$, either from the right or from the left.

Horizontal Asymptotes: The line y = b is a horizontal asymptote of the graph of f(x) if $f(x) \to b$, as $x \to \infty$ or $-\infty$

Identify the vertical and horizontal asymptotes.

14) 14) ___________



Vertical Asymptotes: x = −5 and x = 5
Horizontal Asymptote: y = 3

Determining Vertical Asymptotes: Set the denominator to zero, and solve for x. The values of x determine the locations of the vertical asymptotes. A rational function may have no vertical asymptotes, one vertical asymptote, or several vertical asymptotes. The graph of a rational function never intersects a vertical asymptote. Dashed lines are used to show asymptotes.

Determining Horizontal Asymptotes:

a. If the numerator has lower degree than the denominator, then the x–axis is the horizontal asymptote.

b. If the numerator and the denominator have the same degree, then the horizontal asymptote is $y = \dfrac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$.

Many, but not all, rational functions have horizontal asymptotes.

Find the domain of the function and identify any vertical and horizontal asymptotes. Sketch the graph of the function.

15) $f(x) = \dfrac{2}{x - 1}$ 15) ___________

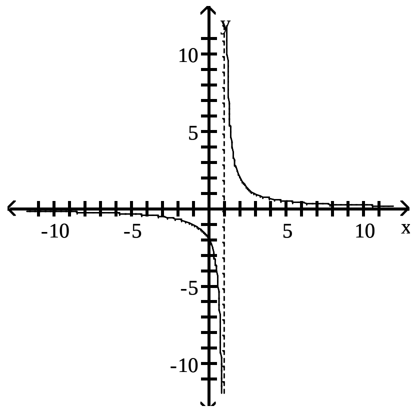
Set $x - 1 = 0 \rightarrow x = 1$ Domain: $(-\infty, 1) \cup (1, \infty)$

Vertical Asymptote: $x = 1$ Horizontal Asymptote: x–axis

y–intercept: $x = 0 \rightarrow y = -2$

x–intercept: $y = 0 \rightarrow$ no solutions

x	2	–1	0
y	2	–1	–2



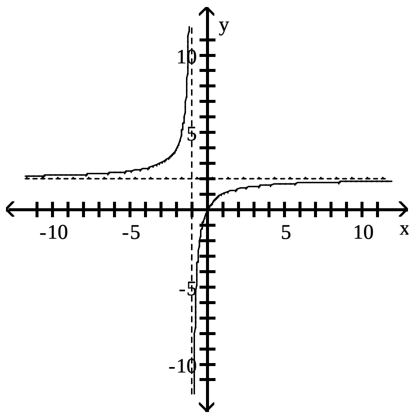
Find the domain of the function and identify any vertical and horizontal asymptotes. Sketch the graph of the function.

16) $f(x) = \dfrac{2x}{x + 1}$ 16) ___________

Set $x + 1 = 0 \rightarrow x = -1$ Domain: $(-\infty, -1) \cup (-1, \infty)$

Vertical Asymptote: $x = -1$ Horizontal Asymptote: $y = \dfrac{2}{1} = 2$

x	−2	1	0
y	4	1	0



Find the domain of the function and identify any vertical and horizontal asymptotes. Sketch the graph of the function

17) $f(x) = \dfrac{x - 2}{x^2 - x - 20}$ 17) ___________

Set $x^2 - x - 20 = 0 \;\rightarrow\; (x + 4)(x - 5) = 0$

$x + 4 = 0 \;\rightarrow\; x = -4 \quad ; \quad x - 5 = 0 \;\rightarrow\; x = 5$

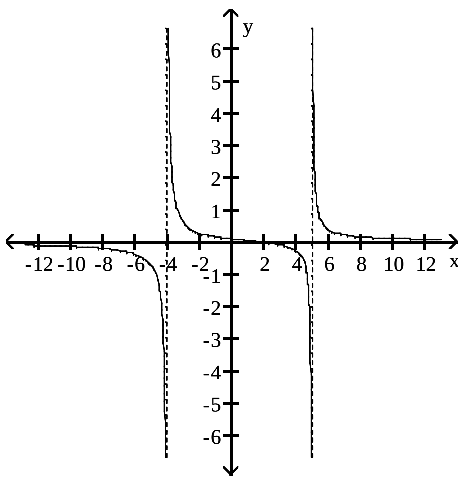
The domain is all real numbers except $x = -4$ and $x = 5$, also written as $(-\infty, -4) \cup (-4, 5) \cup (5, \infty)$.

Vertical Asymptotes: $x = -4$ and $x = 5$

Horizontal Asymptote: x-axis

y-intercept: $x = 0 \rightarrow y = \dfrac{1}{10}$ x-intercept: $y = 0 \rightarrow x = 2$

x	-5	-2	4	6
y	-7/10	2/7	-1/4	2/5



18) $f(x) = \frac{5x + 20}{x^2 + x - 12}$ 18) ____________

Set $x^2 + x - 12 = 0 \rightarrow (x + 4)(x - 3) = 0$

$x + 4 = 0 \rightarrow x = -4$; $x - 3 = 0 \rightarrow x = 3$

The domain is all real numbers except $x = -4$ and $x = 3$, also written as $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$.

$f(x) = \frac{5(x + 4)}{(x + 4)(x - 3)} = \frac{5}{x - 3}$

The graph has a hole corresponding to $x = -4$

Vertical Asymptote: $x = 3$

Horizontal Asymptote: x-axis

Some rational functions have a slant(an oblique) asymptote.
Oblique Asymptotes: If the degree of the numerator is one more than the degree of the denominator, then there will be an oblique asymptote. To find it, divide the numerator by the denominator, and disregard the remainder. Set the quotient equal to y to obtain the equation of the oblique asymptote.

Find the domain of the function and identify any vertical, horizontal, and oblique asymptotes.

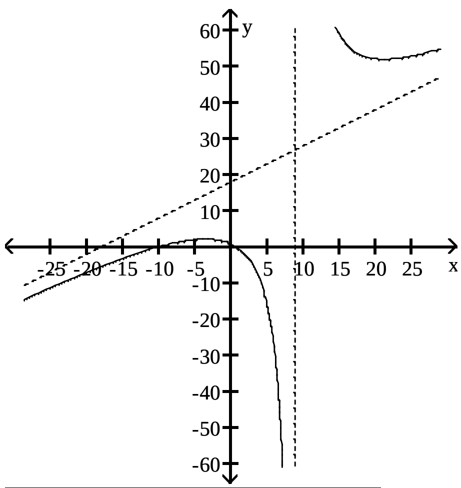
19) $f(x) = \dfrac{x^2 + 9x - 7}{x - 9}$ 19) ___________

Set $x - 9 = 0 \rightarrow x = 9$ Domain: $(-\infty, 9) \cup (9, \infty)$

Vertical Asymptote: $x = 9$ Horizontal Asymptote: none

$$\begin{array}{r|rrr} 9 & 1 & 9 & -7 \\ & & 9 & 162 \\ \hline & 1 & 18 & 155 \end{array}$$

Oblique Asymptote: $y = x + 18$



x	20	5
y	573/11	-63/4

Find the x-intercepts (if any) of the rational function.

20) $f(x) = 2 + \dfrac{-14}{x^2 - 2}$ 20) ___________

Set $y = 0 \rightarrow 2 + \dfrac{-14}{x^2 - 2} = 0 \rightarrow \dfrac{-14}{x^2 - 2} = -2 \rightarrow -2x^2 + 4 = -14$

$\rightarrow -2x^2 = -18 \rightarrow x^2 = 9 \rightarrow x = \pm\sqrt{9} = \pm 3$

2.6 Exercises pg377 (3, 23, 37, 57, 81) (5, 27, 43, 63, 85)

7.3 Partial Fractions

The Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$; where Q(x) has distinct linear factors

$$= \frac{A_1}{(a_1x + b_1)} + \frac{A_2}{(a_2x + b_2)} + \cdots + \frac{A_n}{(a_nx + b_n)}$$

Find the partial fraction decomposition for the expression.

21) $\frac{x + 14}{x^2 - 2x - 8} = \frac{x + 14}{(x - 4)(x + 2)}$ 21) ___________

Step 1: Set up the partial fraction decomposition with the unknown constants.

$$\frac{x + 14}{(x - 4)(x + 2)} = \frac{A}{(x - 4)} + \frac{B}{(x + 2)}$$

Step 2: Multiply both sides of the equation by the least common denominator : $(x - 4)(x + 2)$. We obtain;

$x + 14 = A(x + 2) + B(x - 4)$

Step 3: Simplify the right side of the equation.

$x + 14 = Ax + 2A + Bx - 4B$

Step 4: Write both sides in descending powers.

$x + 14 = Ax + Bx + 2A - 4B$

$x + 14 = (A + B)x + 2A - 4B$

Step 5: Equate coefficients of like powers of x, and equate constant terms.

We obtain: $A + B = 1$ and $2A - 4B = 14$

Step 6: Solve the system of equations for A and B.

$A + B = 1 \rightarrow 4(A + B = 1) \rightarrow 4A + 4B = 4$

$2A - 4B = 14 \rightarrow 2A - 4B = 14$

Adding the two equations, we get $6A = 18 \rightarrow A = 3$

$B = 1 - A = 1 - 3 = -2$

So, $\frac{x + 14}{(x - 4)(x + 2)} = \frac{3}{(x - 4)} + \frac{-2}{(x + 2)}$

The Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$; where Q(x) has distinct linear and quadratic factors

$$\frac{P(x)}{(a_1x + b_1)(a_2x^2 + b_2x + c_2)} = \frac{A_1}{(a_1x + b_1)} + \frac{A_2x + B_2}{(a_2x^2 + b_2x + c_2)}$$

Find the partial fraction decomposition for the expression.

22) $\frac{7x^2 - 7x + 23}{(x - 3)(x^2 + 4)}$ 22) ___________

Step 1: Set up the partial fraction decomposition with the unknown constants.

$$\frac{7x^2 - 7x + 23}{(x - 3)(x^2 + 4)} = \frac{A}{(x - 3)} + \frac{Bx + C}{(x^2 + 4)}$$

Step 2: Multiply both sides of the equation by the least common denominator :$(x - 3)(x^2 + 4)$. We obtain;

$7x^2 - 7x + 23 = A(x^2 + 4) + (Bx + C)(x - 3)$

Step 3: Simplify the right side of the equation.

$7x^2 - 7x + 23 = Ax^2 + 4A + Bx^2 - 3Bx + Cx - 3C$

Step 4: Write both sides in descending powers.

$7x^2 - 7x + 23 = Ax^2 + Bx^2 - 3Bx + Cx + 4A - 3C$

$7x^2 - 7x + 23 = (A + B)x^2 + (-3B + C)x + 4A - 3C$

Step 5: Equate coefficients of like powers of x, and equate constant terms.

We obtain: $A + B = 7$; $-3B + C = -7$; $4A - 3C = 23$

Step 6: Solve the system of equations for A, B, and C.

$A + B = 7$

$3(-3B + C = -7) \rightarrow -9B + 3C = -21$ (equation 2)

$4A - 3C = 23$ (equation 3)

Adding equations (2) and (3), we get $4A - 9B = 2$ (equation 4)

$9(A + B = 7) \rightarrow 9A + 9B = 63$ (equation 5)

Adding equations (4) and (5), we get $13A = 65 \rightarrow A = 5$

$B = 7 - A = 7 - 5 = 2$

$C = 3B - 7 = 6 - 7 = -1$

So, $\frac{7x^2 - 7x + 23}{(x - 3)(x^2 + 4)} = \frac{A}{(x - 3)} + \frac{Bx + C}{(x^2 + 4)} = \frac{5}{(x - 3)} + \frac{2x - 1}{(x^2 + 4)}$

7.3 Exercises pg 809 (11, 30) (12, 29)